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# Confining properties of Abelian-projected theories

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**Abstract.** Representations of the Abelian-projected SU(2) and SU(3) gluodynamics in terms of the magnetic monopole currents are derived. Besides the quadratic part, the obtained effective actions contain interactions of these currents with the world-sheets of electric strings in four dimensions (4D) or electric vortex lines in three dimensions (3D). Next, we illustrate that 3D compact QED is a small gauge-boson mass limit of a 3D Abelian Higgs model with external monopoles and give a physical interpretation of the confining string theory as the integral over the monopole densities. Finally, we derive the bilocal field-strength correlator in the weak-field limit of 3D compact QED, which turns out to be in line with the one predicted by the stochastic vacuum model.

## 1 Introduction

During the last several years, there has appeared a vast amount of papers devoted to the description of confinement in Abelian-projected theories [1] (see, e.g., [2–14] and references therein). The main goal of most of these papers is a derivation of the so-called string representation of such theories, i.e., a reformulation of their partition functions in terms of the integral over the world-sheets of the Abrikosov–Nielsen–Olesen (ANO) strings [15] with a certain nonlocal action (i.e., one that depends on a relative distance between two points in the target space). Such a representation then enables one to get the coupling constants of the corresponding string theory, including higher-order derivative terms, and to evaluate correlators of the dual field-strength tensors [12,13], which play a major role in the so-called stochastic vacuum model (SVM) [16,17]. In the case where there are no external quarks in the underlying non-Abelian theory, the corresponding Abelian-projected theory is some kind of a dual Abelian Higgs model with magnetic Higgs fields, which describe the condensates of monopole Cooper pairs. This model possesses classical solutions, which in four dimensions (4D) are just the electric ANO strings; in three dimensions (3D), they are simply closed electric vortex lines [8]. It is therefore intuitively clear that there should exist some interaction between magnetic monopoles and ANO strings (vortex lines).

It is the first aim of the present paper to demonstrate that such an interaction really exists. To this end, we find it useful to derive the representations of SU(2) and SU(3) Abelian-projected theories directly in terms of monopole currents, which can be done by virtue of the so-called path-integral duality transformation [7,8]. Then it turns out that the resulting effective actions for monopole currents contain, besides the above-mentioned interaction term, also the free part quadratic in the currents. In the so-obtained monopole effective action, all the interactions in the ensemble of monopoles emerging after the Abelian projection are thus emphasized. Namely, the above-mentioned quadratic part gives rise to the Biot–Savart interaction between monopoles as well as to their gauged kinetic energy, whereas the topological interaction term of monopoles with strings or vortices is of the Gauss linking number type, thus reflecting a nontrivial linking between these objects.

Another theory, known for an even longer time, and which allows for an analytic description of confinement, is 3D compact QED [8,18]. In this model, confinement occurs due to Debye screening in the monopole gas. The problem of string representation in this model has been addressed in [4,5], where it has been argued that the desired string theory is formulated in terms of a massive antisymmetric tensor field [19] interacting with the string world-sheet. Such a tensor field is an integer-valued gauge field, whose sources are monopoles. In a phase where monopoles are prolific, the integer-valued nature of this field can be forgotten, and it can be approximated by a continuous-valued field. The latter one is often referred to as the Kalb–Ramond field [19]. The complete form of the action of this field has turned out to be quite a nonlinear one; this was found recently in [20]. After that, the resulting theory, which is usually referred to as the confining string theory, has undergone intensive developments [21]. Notice that the weak-field effective action of this theory appears to have the linear form of the massive Kalb–

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Ramond field action, coinciding with the one of the dual version of the Abelian Higgs model (AHM) in the London limit. This reflects the fact that there should exist a correspondence between the compact QED and AHM direct formulations as well.

The second aim of the present paper is to illustrate the above to be the case, namely that 3D compact QED is nothing more than the small gauge-boson mass limit of 3D AHM with external monopoles. In addition, we shall revisit the compact QED and confining string theories. Specifically, we shall demonstrate that the latter is simply the integral over the monopole densities. This observation gives a physical interpretation to the Kalb–Ramond field as a sum of the monopole and photon field-strength tensors. Finally, we evaluate field correlators in the weak-field limit of 3D compact QED. We argue that their large-distance asymptotic behaviours are in line with the ones following from the general concepts of SVM and observed in the lattice experiments [22]. This result gives a field-theoretical status to SVM and yields a new viewpoint on the confinement phenomenon in compact QED.

The organization of the paper is as follows. In the next section, we study the topic of representation of Abelian-projected theories in terms of magnetic monopole currents. In Sect. 3, we revisit 3D compact QED and its string representation, after which it is demonstrated how this theory can be obtained by a limiting procedure from 3D AHM with monopoles. In the appendix, we outline some details of the path-integral duality transformation.

## 2 Representation of the Abelian-projected theories in terms of monopole currents

Let us start with the 4D SU(2) Abelian-projected gluodynamics, which is argued to be just the dual Abelian Higgs model (DAHM), whose partition function in the London limit has the form

$$\mathcal{Z}_{\rm 4D\,DAHM} = \int DB_\mu D\bar\theta D\theta \exp\left\{-\int \mathrm{d}^4x\left[\frac{1}{4}F_{\mu\nu}^2 + \frac{\eta^2}{2}\left(\partial_\mu\left(\theta+\bar\theta\right) - 2gB_\mu\right)^2\right]\right\}. \qquad (1)$$

Here, $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ is a field-strength tensor of the dual vector potential $B_\mu$, and $g$ is a magnetic coupling constant ($2g$ is the magnetic charge of the monopole Cooper pair). Next, $\eta$ denotes the v.e.v. of the magnetic Higgs field, whose phase has been written as a sum over a multivalued part $\bar\theta$ and a single-valued fluctuation $\theta$ around it. The multivalued field $\bar\theta$ describes all possible electric string configurations[1]. In what follows, we shall restrict ourselves to studying a given one of them. There exists a one-to-one correspondence between the field $\bar\theta$ and the world-sheet coordinate of the ANO string (see (2) below). Note that in this respect, such a correspondence enables us to understand the symbol of the integration measure $D\bar\theta$ as a formal prescription for an integral over string world-sheets (cf. notations in [7,9,12,13])[2].

The above-mentioned correspondence between the multivalued field $\bar\theta$ and the (closed) world-sheet $\Sigma$ of the electric ANO string has the form

$$\varepsilon_{\mu\nu\lambda\rho}\partial_\lambda\partial_\rho\bar\theta(x) = 2\pi\Sigma_{\mu\nu}(x) \equiv 2\pi\int_\Sigma \mathrm{d}\sigma_{\mu\nu}(x(\xi))\delta(x-x(\xi)), \qquad (2)$$

where $\Sigma_{\mu\nu}$ is usually referred to as a vorticity tensor current [7], and $\xi = \left(\xi^1,\xi^2\right)$ stands for the two-dimensional coordinate. Notice that (2) is nothing more than the covariantized version of the Stokes theorem for the vector field $\partial_\mu\bar\theta$. It is also worth noting that the model (1) is actually a continuum version of the corresponding lattice model [8]. In the latter case, $\partial_\mu\bar\theta$ is an integer-valued field as well as the Kalb–Ramond field $h_{\mu\nu}$ (see (3) below). Only in the phase where monopoles become prolific can these fields be replaced by the continuous-valued ones, after which we arrive at the model (1).

Performing now the path-integral duality transformation [5,7,8,12], we obtain the following representation for the partition function (1) (see Appendix A):

$$\mathcal{Z}_{\rm 4D\,DAHM} = \int DA_\mu Dx_\mu(\xi)Dh_{\mu\nu} \times\exp\left\{-\int \mathrm{d}^4x\left[\frac{1}{12\eta^2}H_{\mu\nu\lambda}^2 - \mathrm{i}\pi h_{\mu\nu}\Sigma_{\mu\nu} + \left(gh_{\mu\nu}+\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2\right]\right\}. \qquad (3)$$

In (3), $A_\mu$ is the usual gauge field dual to the vector potential $B_\mu$, and $H_{\mu\nu\lambda} \equiv \partial_\mu h_{\nu\lambda} + \partial_\lambda h_{\mu\nu} + \partial_\nu h_{\lambda\mu}$ is the field strength tensor of a massive antisymmetric tensor field $h_{\mu\nu}$ (the so-called Kalb–Ramond field [19]). This antisymmetric spin-1 tensor field describes a massive dual vector boson. Thus, the path-integral duality transformation is just a way of getting a coupling of this boson to a string world-sheet, rather than to a world-line (as takes place in the usual case of the Wilson loop). In particular, carrying out in (3) the Gaussian integration over the Kalb–Ramond field, one gets a representation of the partition function in terms of an interaction of the elements of the world-sheet $\Sigma$, mediated by the propagator of this field [12]. In what follows, we shall derive another useful representation for the partition function, expressing it directly in terms of magnetic monopole currents.

Notice that according to the equation of motion for the field $A_\mu$, the absence of external electric currents is expressed by the equation $\partial_\mu\mathcal{F}_{\mu\nu} = 0$, where $\mathcal{F}_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + gh_{\mu\nu}$. Regarding $\mathcal{F}_{\mu\nu}$ as a full electromagnetic field-strength tensor, one can write for it the corresponding Bianchi identity modified by the monopoles, $\partial_\mu\tilde{\mathcal{F}}_{\mu\nu} = g\partial_\mu\tilde h_{\mu\nu}$. This identity means that the monopole current

[1] In [8], the gradient of this field has been called the vortex gauge field $\theta_\mu^{\rm v}$.

[2] Clearly, the path-integral measure over multivalued fields in the usual sense of discretized space-time is not defined.

can be written in terms of the Kalb–Ramond field $h_{\mu\nu}$ as

$$j_\mu = g\partial_\nu \tilde{h}_{\nu\mu}, \quad (4)$$

which manifests its conservation.

It is also instructive to write down the equation of motion for the Kalb–Ramond field in terms of the introduced full electromagnetic field-strength tensor. This equation has the form $\mathcal{F}_{\nu\lambda} = (g/m^2)\partial_\mu H_{\mu\nu\lambda} + (\mathrm{i}\pi/2g)\Sigma_{\nu\lambda}$, where $m = 2g\eta$ stands for the mass of the dual gauge boson (equal to the mass of the Kalb–Ramond field). By virtue of conservation of the vorticity tensor current for the closed string world-sheets, $\partial_\mu \Sigma_{\mu\nu} = 0$, this equation again yields the condition of absence of external electric currents, $\partial_\mu \mathcal{F}_{\mu\nu} = 0$.

Let us now turn to a derivation of the monopole current representation for the partition function of the DAHM. To this end, we shall first solve the equation $(g/2)\varepsilon_{\mu\nu\lambda\rho}\partial_\nu h_{\lambda\rho} = -j_\mu$ w.r.t. $h_{\mu\nu}$:

$$h_{\mu\nu}(x) = -\frac{1}{2\pi^2 g}\varepsilon_{\mu\nu\lambda\rho}\int \mathrm{d}^4y \frac{(x-y)_\lambda}{|x-y|^4} j_\rho(y).$$

Next, we get the following expressions for various terms on the right-hand side (R.H.S.) of (3)

$$H^2_{\mu\nu\lambda} = \frac{6}{g^2} j_\mu^2,$$
$$\int \mathrm{d}^4x h^2_{\mu\nu} = \frac{1}{2\pi^2 g^2}\int \mathrm{d}^4x \mathrm{d}^4y j_\mu(x)\frac{1}{(x-y)^2} j_\mu(y).$$

Bringing all this together and performing in (3) the so-called monopole gauge transformation [5] $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$, with the gauge function $\Lambda_\mu = -(1/g)A_\mu$, which eliminates the field $A_\mu$, we finally arrive at the desired monopole current representation, which has the form

$$\mathcal{Z}_{\mathrm{4D\,DAHM}} = \int Dx_\mu(\xi) Dh_{\mu\nu} \times \exp\left\{-\frac{1}{2\pi^2}\int \mathrm{d}^4x \mathrm{d}^4y j_\mu(x)\frac{1}{(x-y)^2} j_\mu(y) - \frac{2}{m^2}\int \mathrm{d}^4x j_\mu^2 + \frac{2\pi\mathrm{i}}{g} S_{\mathrm{int.}}(\Sigma, j_\mu)\right\}. \quad (5)$$

The first term in the exponent on the R.H.S. of (5) has the form of the Biot–Savart energy of the electric field generated by monopole currents [8], the second term corresponds to the (gauged) kinetic energy of Cooper pairs, and the term

$$S_{\mathrm{int.}}(\Sigma, j_\mu) = \frac{1}{4\pi^2}\varepsilon_{\mu\nu\lambda\rho}\int \mathrm{d}^4x \mathrm{d}^4y j_\mu(x)\frac{(y-x)_\nu}{|y-x|^4}\Sigma_{\lambda\rho}(y) \quad (6)$$

describes the interaction of the string world-sheet with the monopole current $j_\mu$. This interaction can obviously be rewritten in the form $S_{\mathrm{int.}} = \int \mathrm{d}^4x j_\mu H^{\mathrm{str.}}_\mu$, where $H^{\mathrm{str.}}_\mu$ is the four-dimensional analog of the magnetic induction, produced by the electric string according to the equation

$$\varepsilon_{\mu\nu\lambda\rho}\partial_\lambda H^{\mathrm{str.}}_\rho = \Sigma_{\mu\nu}. \quad (7)$$

Notice that if one includes an additional current describing an external monopole,

$$j^{\mathrm{ext.}}_\mu(x) = g\oint_\Gamma dx_\mu(\tau)\delta(x - x(\tau)), \quad (8)$$

there arises, among other things, an interaction term (6), which in this case takes the form $S_{\mathrm{int.}} = g\hat{L}(\Sigma, \Gamma)$, where $\hat{L}(\Sigma, \Gamma)$ is simply the Gauss linking number of the world-sheet $\Sigma$ with the contour[3] $\Gamma$.

Clearly, the functional integral over the Kalb–Ramond field in (5) has to be evaluated at the saddle point

$$h^{\mathrm{s.p.}}_{\mu\nu}(x) = \frac{\mathrm{i}g\eta^3}{\pi}\int_\Sigma \mathrm{d}\sigma_{\mu\nu}(x(\xi))\frac{K_1(m|x - x(\xi)|)}{|x - x(\xi)|},$$

where from here on, $K_n$, $n = 0, 1, 2, \cdots$ stands for the modified Bessel function. By virtue of (4), the monopole current can then be expressed via the string world-sheet $\Sigma$ as follows:

$$j_\mu(x) = \frac{\mathrm{i}m^2\eta}{8\pi}\varepsilon_{\mu\nu\lambda\rho}\int_\Sigma \mathrm{d}\sigma_{\lambda\rho}(x(\xi))\frac{(x - x(\xi))_\nu}{(x - x(\xi))^2} \times\left\{\frac{K_1(m|x - x(\xi)|)}{|x - x(\xi)|} + \frac{m}{2}\left[K_0(m|x - x(\xi)|) + K_2(m|x - x(\xi)|)\right]\right\}.$$

It is straightforward to extend the above analysis to the case of the effective dual theory of Abelian-projected SU(3) gluodynamics [2], which is nothing more than DAHM with the $[\mathrm{U}(1)]^2$ gauge invariance. In that case, the partition function (1) is replaced by

$$\mathcal{Z}_{\mathrm{SU(3)}} = \int D\mathbf{B}_\mu D\bar{\theta}_a D\theta_a \delta\left(\sum_{a=1}^{3}\left(\theta_a + \bar{\theta}_a\right)\right) \times \exp\left\{-\int \mathrm{d}^4x\left[\frac{1}{4}\mathbf{F}^2_{\mu\nu} + \frac{\eta^2}{2}\sum_{a=1}^{3}\left(\partial_\mu\left(\theta_a + \bar{\theta}_a\right) - g\boldsymbol{\varepsilon}_a\mathbf{B}_\mu\right)^2\right]\right\} \quad (9)$$

where $\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{B}_\nu - \partial_\nu \mathbf{B}_\mu$ stands for the field-strength tensor of the Abelian vector potential $\mathbf{B}_\mu \equiv \left(B^3_\mu, B^8_\mu\right)$, dual to the usual vector potential $\mathbf{A}_\mu \equiv \left(A^3_\mu, A^8_\mu\right)$. Next, on the R.H.S. of (9),

$$\boldsymbol{\varepsilon}_1 = (1, 0),\ \boldsymbol{\varepsilon}_2 = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right),\ \boldsymbol{\varepsilon}_3 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

[3] Topological interactions of this kind are sometimes interpreted as 4D analogs of the Aharonov–Bohm effect. In particular, this interaction, albeit for the current of an external electrically charged particle with the string world-sheet, emerges in the string representation for the Wilson loop of this particle in AHM [9].

denote the so-called root vectors, and the constraint $\sum_{a=1}^{3}\left(\theta_a+\bar{\theta}_a\right)=0$ is due to the fact that the unitary group under study is special. The multivalued fields $\bar{\theta}_a$ are related to the world-sheets of the three types of strings as

$$\varepsilon_{\mu\nu\lambda\rho}\partial_\lambda\partial_\rho\bar{\theta}_a(x)=2\pi\Sigma^a_{\mu\nu}(x) \equiv 2\pi\int\limits_{\Sigma_a}d\sigma_{\mu\nu}(x_a(\xi))\delta(x-x_a(\xi)), \quad (10)$$

where $x_a\equiv x^a_\mu(\xi)$ is a four-vector parametrizing the world-sheet $\Sigma_a$.

Performing the path-integral duality transformation of (9) by making use of (10), we obtain (see [13] for the details)

$$\mathcal{Z}_{SU(3)}=\int Dx^a_\mu(\xi)\delta\left(\sum_{a=1}^{3}\Sigma^a_{\mu\nu}\right)DA^a_\mu Dh^a_{\mu\nu} \times\exp\left\{-\int d^4x\left[\frac{1}{12\eta^2}\left(H^a_{\mu\nu\lambda}\right)^2-i\pi h^a_{\mu\nu}\Sigma^a_{\mu\nu} +\left(g\frac{\sqrt{3}}{2\sqrt{2}}h^a_{\mu\nu}+\partial_\mu A^a_\nu-\partial_\nu A^a_\mu\right)^2\right]\right\}, \quad (11)$$

where $A^a_\mu\equiv\boldsymbol{\varepsilon}_a\mathbf{A}_\mu$. (11) means that the three monopole currents can be expressed in terms of three Kalb–Ramond fields as $j^a_\mu=g(\sqrt{3}/2\sqrt{2})\partial_\nu\tilde{h}^a_{\nu\mu}$ (cf. (4)). Finally, rewriting (11) via these currents and resolving the constraint $\sum_{a=1}^{3}\Sigma^a_{\mu\nu}=0$ by integrating over one of the world-sheets (for concreteness, $x^3_\mu(\xi)$), we obtain

$$\mathcal{Z}_{SU(3)}=\int Dx^1_\mu(\xi)Dx^2_\mu(\xi)Dh^a_{\mu\nu} \times\exp\left\{-\frac{1}{2\pi^2}\int d^4xd^4y j^a_\mu(x)\frac{1}{(x-y)^2}j^a_\mu(y) -\frac{2}{m_B^2}\int d^4x\left(j^a_\mu\right)^2 +4\pi i\sqrt{\frac{2}{3}}\frac{1}{g}\left[S_{\rm int.}\left(\Sigma^1,j^1_\mu\right)+S_{\rm int.}\left(\Sigma^2,j^2_\mu\right) -S_{\rm int.}\left(\Sigma^1,j^3_\mu\right)-S_{\rm int.}\left(\Sigma^2,j^3_\mu\right)\right]\right\}, \quad (12)$$

where $m_B=\sqrt{3/2}g\eta$ stands for the masses of the fields $B^3_\mu$ and $B^8_\mu$, which they acquire due to the Higgs mechanism. Equation (12) is the desired representation for the partition function of the Abelian-projected SU(3) gluodynamics in terms of three monopole currents, which should be evaluated at the saddle point. The terms in square brackets on its R.H.S. yield an interference between various possibilities of the interaction between the string world-sheets and monopole currents in this model.

For illustration, let us establish a correspondence of the above results to the 3D ones. Namely, let us derive a 3D analog of (5), i.e., find a representation in terms of the monopole currents of the dual Ginzburg–Landau model. There, (2) is replaced by [8]:

$$\varepsilon_{\mu\nu\lambda}\partial_\nu\partial_\lambda\bar{\theta}\,(\mathbf{x})=2\pi\delta_\mu\,(\mathbf{x})\,. \quad (13)$$

Here on the R.H.S. stands the so-called vortex density with

$$\delta_\mu\,(\mathbf{x})\equiv\int\limits_L dy_\mu(\tau)\delta\,(\mathbf{x}-\mathbf{y}(\tau)) \quad (14)$$

being the transverse $\delta$ function defined w.r.t. the electric vortex line $L$, parametrized by the vector $\mathbf{y}(\tau)$. This line is closed in the case under study, i.e., in the absence of external quarks, which means that $\partial_\mu\delta_\mu=0$. Performing again (by virtue of (13)) the path-integral duality transformation of the partition function (1) with the 3D action, we get the following representation:

$$\mathcal{Z}_{\rm 3D\,DAHM}=\int D\varphi Dy_\mu(\tau)Dh_\mu \times\exp\left\{-\int d^3x\left[\frac{1}{4\eta^2}\left(\partial_\mu h_\nu-\partial_\nu h_\mu\right)^2 -2\pi i h_\mu\delta_\mu+\left(g\sqrt{2}h_\mu+\partial_\mu\varphi\right)^2\right]\right\}. \quad (15)$$

Notice that the Kalb–Ramond field has now reduced to a massive one-form field $h_\mu$ with the mass $m=2g\eta$, and the $A_\mu$ field has reduced to a scalar $\varphi$. As in the 4D case, the field $\mathcal{E}_\mu\equiv g\sqrt{2}h_\mu+\partial_\mu\varphi$ can be regarded as a full electric field, defined via the full dual electromagnetic field-strength tensor as $\mathcal{E}_\mu=(1/2)\varepsilon_{\mu\nu\lambda}\mathcal{F}_{\nu\lambda}$. The absence of external quarks is now expressed by the equation $\partial_\mu\mathcal{E}_\mu=0$, following from the equation of motion for the field $\varphi$. Correspondingly, the monopole currents are defined as $j_\nu=\partial_\mu\mathcal{F}_{\mu\nu}=g\sqrt{2}\varepsilon_{\mu\nu\lambda}\partial_\mu h_\lambda$ and are manifestly conserved. Notice also that the condition of closeness of the vortex lines, $\partial_\mu\delta_\mu=0$, unambiguously exhibits itself as a condition of absence of external quarks, $\partial_\mu\mathcal{E}_\mu=0$, by virtue of the equation of motion for the field $h_\mu$, which can be written in the form $\mathcal{E}_\mu=(1/g\sqrt{2})\left[(1/2\eta^2)\partial_\nu\left(\partial_\nu h_\mu-\partial_\mu h_\nu\right)+i\pi\delta_\mu\right]$.

Next, after performing the monopole gauge transformation $h_\mu\to h_\mu+\partial_\mu\gamma$ with the gauge function $\gamma=-(1/g\sqrt{2})\varphi$, the field $\varphi$ drops out. Expressing $h_\mu$ via $j_\mu$,

$$h_\mu\,(\mathbf{x})=-\frac{1}{4\sqrt{2}\pi g}\varepsilon_{\mu\nu\lambda}\frac{\partial}{\partial x_\nu}\int d^3y\frac{j_\lambda\,(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}$$

and substituting this expression into the R.H.S. of (15), we finally arrive at the desired representation for the partition function of 3D DAHM in terms of the monopole currents:

$$\mathcal{Z}_{\rm 3D\,DAHM}=\int Dy_\mu(\tau)Dh_\mu \times\exp\left\{-\left[\frac{1}{4\pi}\int d^3xd^3y j_\mu\,(\mathbf{x})\,\frac{1}{|\mathbf{x}-\mathbf{y}|}j_\mu\,(\mathbf{y})\right.\right.$$

$$+\frac{1}{m^2}\int \mathrm{d}^3 x j_\mu^2 + \frac{\sqrt{2}\pi\mathrm{i}}{g} S_{\text{int.}}(L, j_\mu)\Bigg]\Bigg\}. \quad (16)$$

The interaction term of the electric vortex line with the monopole current now takes the form

$$S_{\text{int.}}(L, j_\mu) = \frac{1}{4\pi}\varepsilon_{\mu\nu\lambda}\int \mathrm{d}^3x\mathrm{d}^3y j_\mu(\mathbf{x})\frac{(\mathbf{y}-\mathbf{x})_\nu}{|\mathbf{y}-\mathbf{x}|^3}\delta_\lambda(\mathbf{y}).$$

This interaction term can be again rewritten as $S_{\text{int.}} = \int \mathrm{d}^3 x j_\mu H_\mu^{\text{vor.}}$, where the magnetic induction, generated by the electric vortex line, obeys the equation $\varepsilon_{\mu\nu\lambda}\partial_\nu H_\lambda^{\text{vor.}} = \delta_\mu$. In the particular case, when one introduces an external current of the form (8), there emerges a term $S_{\text{int.}} = g\hat{L}(L,\Gamma)$ with $\hat{L}(L,\Gamma)$ standing for the Gauss linking number of the contours $L$ and $\Gamma$. The functional integral over the field $h_\mu$ in (16) should again be evaluated at the saddle point $h_\mu^{\text{s.p.}}$, which is determined by the classical equation of motion, following from (15) after the field $\varphi$ has been gauged away. This saddle point has the form

$$h_\mu^{\text{s.p.}}(\mathbf{x}) = \frac{\mathrm{i}\eta^2}{2}\oint_L \mathrm{d}y_\mu(\tau)\frac{\mathrm{e}^{-m|\mathbf{x}-\mathbf{y}(\tau)|}}{|\mathbf{x}-\mathbf{y}(\tau)|},$$

which yields the following expression for the monopole current:

$$j_\mu(\mathbf{x}) = \frac{\mathrm{i}g\eta^2}{\sqrt{2}}\varepsilon_{\mu\nu\lambda}\oint_L \mathrm{d}y_\lambda(\tau)\frac{(\mathbf{x}-\mathbf{y}(\tau))_\nu}{(\mathbf{x}-\mathbf{y}(\tau))^2} \times\left(m+\frac{1}{|\mathbf{x}-\mathbf{y}(\tau)|}\right)\mathrm{e}^{-m|\mathbf{x}-\mathbf{y}(\tau)|}.$$

In the next section, we shall investigate the relation between the 3D AHM with external monopoles and the 3D compact QED, as well as the string representation of the 3D compact QED itself.

## 3 Vacuum correlators and string representation of 3D compact QED

In this section, we shall revisit 3D compact QED and find its string representation in the form of an integral over the monopole densities. In addition, we shall investigate vacuum correlators in the weak-field limit, and demonstrate the relation of this theory to 3D AHM with monopoles.

The most important feature of 3D *compact* QED, which distinguishes it from the *noncompact* case, is the existence of magnetic monopoles. Their general configuration is the Coulomb gas with the action [18]

$$S_{\text{mon.}} = g^2\sum_{a<b} q_a q_b\left(\Delta^{-1}\right)(\mathbf{z}_a,\mathbf{z}_b) + S_0\sum_a q_a^2, \quad (17)$$

where $\Delta$ is the 3D Laplace operator, and $S_0$ is the action of a single monopole, $S_0 = \text{const.}/e^2$. Here, similarly to [18], we have adopted standard Dirac notations, where $eg = 2\pi n$, restricting ourselves to the monopoles of the minimal charge, i.e., setting $n = 1$. Then the partition function of the grand canonical ensemble of monopoles associated with the action (17) reads

$$\mathcal{Z}_{\text{mon.}} = \sum_{N=0}^{+\infty}\sum_{q_a=\pm1}\frac{\zeta^N}{N!}\prod_{i=0}^{N}\int \mathrm{d}^3 z_i \quad (18)$$
$$\times\exp\left[-\frac{\pi}{2e^2}\int \mathrm{d}^3x\mathrm{d}^3y\rho_{\text{gas}}(\mathbf{x})\frac{1}{|\mathbf{x}-\mathbf{y}|}\rho_{\text{gas}}(\mathbf{y})\right],$$

where $\rho_{\text{gas}}(\mathbf{x}) = \sum_a q_a\delta(\mathbf{x}-\mathbf{z}_a)$ is the monopole density, corresponding to the gas configuration. Here, a single monopole weight $\zeta \propto \exp(-S_0)$ has the dimension of $(\text{mass})^3$; this is usually referred to as fugacity. Notice also that, as usual, we have restricted ourselves to the values $q_a = \pm1$, since at large values of the magnetic coupling constant $g$, monopoles with $|q| > 1$ turn out to be unstable, and tend to dissociate into the monopoles with $|q| = 1$. Later in this section, it will be demonstrated that the limit of a small gauge-boson mass (which takes place, e.g., at large $g$) is just the case when 3D compact QED follows from 3D AHM with external monopoles.

Next, Coulomb interaction can be made local, albeit nonlinear, by introduction of an auxiliary scalar field [18]:

$$\mathcal{Z}_{\text{mon.}} = \int D\chi\exp\left\{-\int \mathrm{d}^3x\left[\frac{1}{2}(\partial_\mu\chi)^2 - 2\zeta\cos(g\chi)\right]\right\}. \quad (19)$$

The magnetic mass $m = g\sqrt{2\zeta}$ of the field $\chi$, following from the quadratic term in the expansion of the cosine on the R.H.S. of (19), is due to the Debye screening in the monopole plasma. The next, quartic, term of the expansion determines the coupling constant of the diagrammatic expansion for the monopole gas, which is therefore exponentially small and proportional to $g^4\exp(-\text{const.}g^2)$.

Let us now cast the partition function (19) into the form of an integral over the monopole densities. This can be done by introducing into (18) a unity of the form

$$\int D\rho\delta(\rho(\mathbf{x})-\rho_{\text{gas}}(\mathbf{x}))$$
$$= \int D\rho D\chi\exp\left\{\mathrm{i}g\left[\sum_a q_a\chi(\mathbf{z}_a) - \int \mathrm{d}^3x\chi\rho\right]\right\},$$

where we have omitted the inessential normalization factor. Next, performing the summations in (18), we get

$$\mathcal{Z}_{\text{mon.}} = \int D\rho D\chi\exp\left\{-\frac{\pi}{2e^2}\int \mathrm{d}^3x\mathrm{d}^3y\rho(\mathbf{x})\frac{1}{|\mathbf{x}-\mathbf{y}|}\rho(\mathbf{y}) + \int \mathrm{d}^3x\Big[2\zeta\cos(g\chi) - \mathrm{i}g\chi\rho\Big]\right\}. \quad (20)$$

Finally, integrating over the field $\chi$ by resolving the corresponding saddle-point equation,

$$\sin(g\chi) = -\frac{\mathrm{i}\rho}{2\zeta}, \quad (21)$$

we arrive at the desired representation for the partition function

$$\mathcal{Z}_{\rm mon.} = \int D\rho \exp\left\{-\left[\frac{\pi}{2e^2}\int d^3x d^3y \rho(\mathbf{x}) \times \frac{1}{|\mathbf{x}-\mathbf{y}|}\rho(\mathbf{y}) + V[\rho]\right]\right\}, \quad (22)$$

where

$$V[\rho] = \int d^3x \left\{\rho \ln\left[\frac{\rho}{2\zeta} + \sqrt{1+\left(\frac{\rho}{2\zeta}\right)^2}\right] -2\zeta\sqrt{1+\left(\frac{\rho}{2\zeta}\right)^2}\right\} \quad (23)$$

is the parabolic-type effective monopole potential, whose asymptotic behaviours at $\rho \ll \zeta$ and $\rho \gg \zeta$ read

$$V[\rho] \longrightarrow \int d^3x \left(-2\zeta + \frac{\rho^2}{4\zeta}\right) \quad (24)$$

and

$$V[\rho] \longrightarrow \int d^3x \left[\rho\left(\ln\frac{\rho}{\zeta} - 1\right)\right],$$

respectively. Notice that during the integration over the field $\chi$ in (20), we have chosen only the real branch of the solution to the saddle-point equation (21) and disregarded the complex ones.

The obtained representation for the partition function in terms of the monopole densities can be immediately applied to the calculation of the coefficient function $\mathcal{D}^{\rm mon.}(x^2)$, related to the bilocal correlator of the field-strength tensors as follows [16,17]

$$\langle \mathcal{F}_{\lambda\nu}(\mathbf{x})\mathcal{F}_{\mu\rho}(0)\rangle_{A_\mu,\rho} = \left(\delta_{\lambda\mu}\delta_{\nu\rho} - \delta_{\lambda\rho}\delta_{\nu\mu}\right)\mathcal{D}^{\rm mon.}(x^2) + \frac{1}{2}\left[\partial_\lambda\left(x_\mu\delta_{\nu\rho} - x_\rho\delta_{\nu\mu}\right) + \partial_\nu\left(x_\rho\delta_{\lambda\mu} - x_\mu\delta_{\lambda\rho}\right)\right]\mathcal{D}_1^{\rm full}(x^2), \quad (25)$$

where the average over the monopole densities is defined by the partition function (22), whereas the $A_\mu$-average is defined as

$$\langle ...\rangle_{A_\mu} \equiv \frac{\int DA_\mu (...) \exp\left(-\frac{1}{4e^2}\int d^3x F_{\mu\nu}^2\right)}{\int DA_\mu \exp\left(-\frac{1}{4e^2}\int d^3x F_{\mu\nu}^2\right)}.$$

In (25), $\mathcal{F}_{\mu\nu} = F_{\mu\nu} + F_{\mu\nu}^M$ stands for the full electromagnetic field-strength tensor, which includes also the monopole part

$$F_{\mu\nu}^M(\mathbf{x}) = -\frac{1}{2}\varepsilon_{\mu\nu\lambda}\frac{\partial}{\partial x_\lambda}\int d^3y \frac{\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}.$$

This monopole part yields the R.H.S. of the Bianchi identities modified by the monopoles,

$$\partial_\mu \mathcal{H}_\mu = 2\pi\rho, \quad (26)$$

where $\mathcal{H}_\mu = (1/2)\varepsilon_{\mu\nu\lambda}\mathcal{F}_{\nu\lambda}$ stands for the full magnetic induction. Equations (25) and (26) then lead to the following equation for the function $\mathcal{D}^{\rm mon.}$:

$$\Delta\mathcal{D}^{\rm mon.}(x^2) = -4\pi^2\langle\rho(\mathbf{x})\rho(0)\rangle_\rho, \quad (27)$$

which in fact is a 3D analog of the 4D equation [17]

$$\left(\partial_\mu\partial_\nu - \partial^2\delta_{\mu\nu}\right)\mathcal{D}^{\rm mon.}(x^2) = \langle j_\mu(x) j_\nu(0)\rangle.$$

The correlator standing on the R.H.S. of (27) can be found in the limit of small monopole densities, $\rho \ll \zeta$. By making use of (22) and (24), we obtain

$$\langle\rho(\mathbf{x})\rho(0)\rangle_\rho = -\frac{\zeta}{2\pi}\Delta\frac{e^{-m|\mathbf{x}|}}{|\mathbf{x}|}.$$

Then, demanding that $\mathcal{D}^{\rm mon.}(x^2 \to \infty) \to 0$, we get by the maximum principle for the harmonic functions the desired expression for the function $\mathcal{D}^{\rm mon.}$ in the low-density limit:

$$\mathcal{D}^{\rm mon.}(x^2) = 2\pi\zeta\frac{e^{-m|\mathbf{x}|}}{|\mathbf{x}|}. \quad (28)$$

We see that in the model under study, the correlation length of the vacuum [16,17] $T_g$, i.e., the distance at which the function $\mathcal{D}^{\rm mon.}$ decreases, corresponds to the inverse mass of the field $\chi$, $m^{-1}$ (cf. the case of Abelian-projected theories, studied in [12,13]). The coefficient function $\mathcal{D}_1^{\rm full}(x^2)$ will be derived later on.

Let us now proceed to the problem of string representation of the 3D compact QED. To this end, let us consider an expression for the Wilson loop and try to represent it as an integral over the world-sheets $\Sigma$, bounded by the contour $C$. By virtue of the Stokes theorem, the Wilson loop can be rewritten in the form

$$\langle W(C)\rangle = \left\langle \exp\left(\frac{i}{2}\int_\Sigma d\sigma_{\mu\nu}\mathcal{F}_{\mu\nu}\right)\right\rangle_{A_\mu,\rho} = \left\langle \exp\left(i\int_\Sigma d\sigma_\mu \mathcal{H}_\mu\right)\right\rangle_{A_\mu,\rho} = \langle W(C)\rangle_{A_\mu}\left\langle \exp\left(\frac{i}{2}\int d^3x \rho(\mathbf{x})\eta(\mathbf{x})\right)\right\rangle_\rho, \quad (29)$$

where the free-photon contribution reads

$$\langle W(C)\rangle_{A_\mu} = \left\langle \exp\left(i\oint_C A_\mu dx_\mu\right)\right\rangle_{A_\mu} = \exp\left(-\frac{e^2}{8\pi}\oint_C dx_\mu \oint_C dy_\mu \frac{1}{|\mathbf{x}-\mathbf{y}|}\right). \quad (30)$$

In (29), $d\sigma_\mu \equiv (1/2)\varepsilon_{\mu\nu\lambda}d\sigma_{\nu\lambda}$, and $\eta(\mathbf{x}) = (\partial/\partial x_\mu) \times \int_\Sigma d\sigma_\mu(\mathbf{y})(1/|\mathbf{x}-\mathbf{y}|)$ stands for the solid angle under which the surface $\Sigma$ is seen by an observer at the point $\mathbf{x}$. Notice that due to the Gauss law, in the case when $\Sigma$ is a closed surface surrounding the point $\mathbf{x}$, $\eta(\mathbf{x})$ is equal to $4\pi$, which is the standard result for the total solid angle in 3D.

Equation (29) seems to contain a discrepancy: its left-hand side (L.H.S.) depends only on the contour $C$, whereas the R.H.S. depends on an arbitrary surface $\Sigma$, bounded by $C$. However, this actually turns out not to be a discrepancy, but a key point in the construction of the desired string representation. The resolution of the apparent paradox lies in the observation that during the derivation of the effective monopole potential (23), we have accounted only for the one, namely real, branch of the solution to the saddle-point equation (21). Actually, however, one should sum up over all the (complex-valued) branches of the integrand of the effective potential (23) at every space point $\mathbf{x}$. This requires to replace $V[\rho]$ by

$$V_{\rm total}[\rho] = \sum_{n=-\infty}^{+\infty}\int {\rm d}^3x\left\{\rho\left(\ln\left[\frac{\rho}{2\zeta}+\sqrt{1+\left(\frac{\rho}{2\zeta}\right)^2}\right]+2\pi in\right)-2\zeta\sqrt{1+\left(\frac{\rho}{2\zeta}\right)^2}\right\}.$$

A summation over the branches of the multivalued potential in the expression for the Wilson loop,

$$\langle W(C)\rangle = \langle W(C)\rangle_{A_\mu}\int D\rho\exp\left\{-\left[\frac{\pi}{2e^2}\int {\rm d}^3x{\rm d}^3y\rho(\mathbf{x})\times\frac{1}{|\mathbf{x}-\mathbf{y}|}\rho(\mathbf{y})+V_{\rm total}[\rho]-\frac{\rm i}{2}\int {\rm d}^3x\rho(\mathbf{x})\eta(\mathbf{x})\right]\right\}, \tag{31}$$

thus restores the independence of the choice of the world-sheet. (Notice that from now on we omit an inessential normalization factor, implying everywhere the normalization $\langle W(0)\rangle = 1$.)

It is worth noting that the obtained string representation (31) has been derived, for the first time, in another, more indirect, way [20]. It is therefore instructive to establish a correspondence between the above derivation and the one in that paper.

The main idea of [20] is to calculate the Wilson loop starting with the direct definition of this average in a sense of the partition function (18) of the monopole gas. The corresponding expression has the form

$$\langle W(C)\rangle_{\rm mon.} = \sum_{N=0}^{+\infty}\sum_{q_a=\pm1}\frac{\zeta^N}{N!}\prod_{i=0}^{N}\int {\rm d}^3z_i\times\exp\left[-\frac{\pi}{2e^2}\int {\rm d}^3x{\rm d}^3y\rho_{\rm gas}(\mathbf{x})\frac{1}{|\mathbf{x}-\mathbf{y}|}\rho_{\rm gas}(\mathbf{y})+\frac{\rm i}{2}\int {\rm d}^3x\rho_{\rm gas}(\mathbf{x})\eta(\mathbf{x})\right]$$
$$=\int D\chi\exp\left\{-\int {\rm d}^3x\left[\frac12(\partial_\mu\chi)^2-2\zeta\cos\left(g\chi+\frac{\eta}{2}\right)\right]\right\}$$
$$=\int D\varphi\exp\left\{-\int {\rm d}^3x\left[\frac{e^2}{8\pi^2}\left(\partial_\mu\varphi-\frac12\partial_\mu\eta\right)^2-2\zeta\cos\varphi\right]\right\}, \tag{32}$$

where $\varphi\equiv g\chi+\frac{\eta}{2}$.

Next, one can prove the equality

$$\exp\left[-\frac{e^2}{8\pi}\oint_C {\rm d}x_\mu\oint_C {\rm d}y_\mu\frac{1}{|\mathbf{x}-\mathbf{y}|}-\frac{e^2}{8\pi^2}\int {\rm d}^3x\left(\partial_\mu\varphi-\frac12\partial_\mu\eta\right)^2\right]$$
$$=\int Dh_{\mu\nu}\exp\left[-\int {\rm d}^3x\left({\rm i}\varphi\varepsilon_{\mu\nu\lambda}\partial_\mu h_{\nu\lambda}+g^2h_{\mu\nu}^2-2\pi{\rm i}h_{\mu\nu}\Sigma_{\mu\nu}\right)\right], \tag{33}$$

which makes it possible to represent the contribution of the kinetic term on the R.H.S. of (32) and the free- photon contribution (30) to the Wilson loop as an integral over the Kalb–Ramond field. The only nontrivial point necessary to prove this equality is an expression for the derivative of the solid angle:

$$\partial_\lambda\eta(\mathbf{x}) = \int_\Sigma\left(d\sigma_\mu(\mathbf{y})\frac{\partial}{\partial y_\lambda}-d\sigma_\lambda(\mathbf{y})\frac{\partial}{\partial y_\mu}\right)\frac{\partial}{\partial y_\mu}\frac{1}{|\mathbf{x}-\mathbf{y}|}+\int_\Sigma d\sigma_\lambda(\mathbf{y})\,\Delta\frac{1}{|\mathbf{x}-\mathbf{y}|}. \tag{34}$$

Applying to the first integral on the R.H.S. of (34) the Stokes theorem in the operator form,

$${\rm d}\sigma_\mu\frac{\partial}{\partial y_\lambda}-{\rm d}\sigma_\lambda\frac{\partial}{\partial y_\mu}\longrightarrow\varepsilon_{\mu\lambda\nu}dy_\nu,$$

one finally obtains

$$\partial_\lambda\eta(\mathbf{x}) = \varepsilon_{\lambda\mu\nu}\frac{\partial}{\partial x_\mu}\oint_C {\rm d}y_\nu\frac{1}{|\mathbf{x}-\mathbf{y}|}-4\pi\int_\Sigma {\rm d}\sigma_\lambda(\mathbf{y})\,\delta(\mathbf{x}-\mathbf{y}).$$

Making use of this result and carrying out the Gaussian integral over the field $h_{\mu\nu}$, one can demonstrate that both sides of (33) are equal to

$$\exp\left\{-\frac{e^2}{2}\left[\frac{1}{4\pi^2}\int {\rm d}^3x(\partial_\mu\varphi)^2+\frac1\pi\int_\Sigma {\rm d}\sigma_\mu\partial_\mu\varphi+\int_\Sigma {\rm d}\sigma_\mu(\mathbf{x})\int_\Sigma {\rm d}\sigma_\mu(\mathbf{y})\,\delta(\mathbf{x}-\mathbf{y})\right]\right\}$$

thus proving the validity of this equation.

Substituting now (33) into (32), it is easy to carry out the integral over the field $\varphi$, which has no more kinetic term, in the saddle point approximation. This equation has the same form as (21), with the replacement $\rho \to \varepsilon_{\mu\nu\lambda}\partial_\mu h_{\nu\lambda}$. The resulting expression for the full Wilson loop then takes the form

$$\langle W(C)\rangle = \langle W(C)\rangle_{A_\mu} \langle W(C)\rangle_{\text{mon.}} = \int Dh_{\mu\nu} \exp\left\{-\int d^3x \left(g^2 h_{\mu\nu}^2 + V_{\text{total}}\left[\varepsilon_{\mu\nu\lambda}\partial_\mu h_{\nu\lambda}\right]\right) + 2\pi i \int_\Sigma d\sigma_{\mu\nu} h_{\mu\nu}\right\}, \tag{35}$$

where the world-sheet independence of the R.H.S. is again provided by the summation over the branches of the multivalued action, which is now the action of the Kalb–Ramond field.

Comparing now (31) and (35), we see that the Kalb–Ramond field is indeed related to the monopole density via the equation $\varepsilon_{\mu\nu\lambda}\partial_\mu h_{\nu\lambda} = \rho$. Thus, a conclusion following from the representation of the full Wilson loop in terms of the integral over the Kalb–Ramond field is that this field is simply related to the sum of the photon and monopole field-strength tensors as $h_{\mu\nu} = (1/4\pi)\mathcal{F}_{\mu\nu}$. In the formal language, such a decomposition of the Kalb–Ramond field is just the essence of the Hodge decomposition theorem.

Let us now consider the weak-field limit of (35) and again restrict ourselves to the real branch of the effective potential, i.e., replace $V_{\text{total}}\left[\varepsilon_{\mu\nu\lambda}\partial_\mu h_{\nu\lambda}\right]$ by $V\left[\varepsilon_{\mu\nu\lambda}\partial_\mu h_{\nu\lambda}\right]$. This yields the following expression for the Wilson loop:

$$\langle W(C)\rangle_{\text{weak-field}} = \int Dh_{\mu\nu} \exp\left\{-\int d^3x \left[\frac{1}{6\zeta} H_{\mu\nu\lambda}^2 + g^2 h_{\mu\nu}^2 - 2\pi i h_{\mu\nu}\Sigma_{\mu\nu}\right]\right\}. \tag{36}$$

Notice that the mass of the Kalb–Ramond field resulting from this equation is equal to the mass $m$ of the field $\chi$ from (19).

One can now see that (36) is quite similar to the 3D version of (3) (with the $A_\mu$ field gauged away) that we had in the DAHM case. However, the important difference from the DAHM is that by restricting ourselves to the real branch of the potential, we have violated the surface independence of the R.H.S. of (36). This problem is similar to the one which appears in SVM [16,17], where in the expression for the Wilson loop, written via the non-Abelian Stokes theorem and cumulant expansion, one disregards all the cumulants higher than the bilocal one (the so-called bilocal approximation). There, the surface independence is restored by replacing $\Sigma$ by the surface of the minimal area, $\Sigma_{\text{min.}} = \Sigma_{\text{min.}}[C]$, bounded by the contour $C$. Let us follow this recipe, after which the quantity

$$S_{\text{str.}} = -\ln \langle W(C)\rangle_{\text{weak-field}}\big|_{\Sigma\to\Sigma_{\text{min.}}} \tag{37}$$

can be considered as a weak-field string- effective action of the 3D compact QED.

The integration over the Kalb–Ramond field in (36) is now almost the same as the one of [12] and yields

$$\langle W(C)\rangle_{\text{weak-field}}\big|_{\Sigma\to\Sigma_{\text{min.}}} = \exp\left\{-\frac{1}{8}\int_{\Sigma_{\text{min.}}} d\sigma_{\lambda\nu}(\mathbf{x}) \times \int_{\Sigma_{\text{min.}}} d\sigma_{\mu\rho}(\mathbf{y}) \left\langle \mathcal{F}_{\lambda\nu}(\mathbf{x})\mathcal{F}_{\mu\rho}(\mathbf{y})\right\rangle_{A_\mu,\rho}\right\},$$

which is consistent with the result following directly from the cumulant expansion of (29). Here, the bilocal correlator is defined by (25) with the function $\mathcal{D}^{\text{mon.}}$ given by (28) and $\mathcal{D}_1^{\text{full}} = \mathcal{D}_1^{\text{phot.}} + \mathcal{D}_1^{\text{mon.}}$, where the photon and monopole contributions read

$$\mathcal{D}_1^{\text{phot.}}\left(x^2\right) = \frac{e^2}{2\pi |\mathbf{x}|^3}$$

and

$$\mathcal{D}_1^{\text{mon.}}\left(x^2\right) = \frac{e^2}{4\pi x^2}\left(m + \frac{1}{|\mathbf{x}|}\right) e^{-m|\mathbf{x}|}, \tag{38}$$

respectively. Since the approximation $\rho \ll \zeta$, from which (28) has been derived, is just the weak-field limit in which (36) follows from (35), the coincidence of the function $\mathcal{D}^{\text{mon.}}$ (following from the propagator of the Kalb–Ramond field) with the one of (28) confirms the consistency of our calculations.

Notice that by performing an expansion of the nonlocal string- effective action (37) in powers of the derivatives w.r.t. the world-sheet coordinates $\xi$, one gets the string tension of the Nambu–Goto term and the inverse bare coupling constant of the rigidity term, which are represented by

$$\sigma = \pi^2 \frac{\sqrt{2\zeta}}{g} \quad \text{and} \quad \frac{1}{\alpha_0} = -\frac{\pi^2}{8\sqrt{2\zeta}g^3}, \tag{39}$$

respectively. Similarly to the corresponding quantities in the Abelian-projected SU(2) and SU(3) gluodynamics, found in [12] and [13], both are nonanalytic in $g$, which manifests the nonperturbative nature of string representation of all the three theories. Notice also that the negative sign of $\alpha_0$ is important for the stability of the string world-sheets [21].

We see that the long- and short-distance asymptotic behaviours of the functions (28) and (38) have the same properties as the ones of the corresponding functions in QCD within SVM [22]. Namely, at large distances, both of the functions (28) and (38) decrease exponentially with the correlation length $m^{-1}$, and at such distances, $\mathcal{D}_1^{\text{mon.}} \ll \mathcal{D}^{\text{mon.}}$, due to the pre-exponential factor. In the same time, in the opposite case $|\mathbf{x}| \ll m^{-1}$, the function $\mathcal{D}_1^{\text{mon.}}$ is much larger than the function $\mathcal{D}^{\text{mon.}}$, which also parallels the SVM results. Notice, however, that the short-distance similarity takes place only to the lowest order of perturbation theory in QCD, where its specific non-Abelian properties are not important.

It is also worth noting that the above described asymptotic behaviours of the functions $\mathcal{D}^{\rm mon.}$ and $\mathcal{D}_1^{\rm mon.}$ match those of the corresponding functions, which parametrize the bilocal correlator of the dual field-strength tensors in the DAHM [12]. This similarity, as well as the similarity of (3) and (36), tells us that there should exist some relation between 3D compact QED and 3D AHM. In what follows, we shall demonstrate that such a relation really exists, namely that the 3D compact QED corresponds to the case of the small gauge-boson mass in the London limit of the 3D AHM with monopoles. Let us stress that in 3D, monopoles are considered as particles at rest, contrary to the 4D case, where they are generally treated as world-lines of moving particles. That is why, in order to end up with the 3D compact QED (i.e., the partition function (18) of the monopole gas), one should start with the 3D AHM, with the scalar density $\rho_{\rm gas}$ of external monopoles at rest, rather than with the DAHM. The corresponding partition function has the form

$$\mathcal{Z}_{\rm 3D\,AHM} = \int DA_\mu D\bar\theta D\theta \exp\left\{-\int {\rm d}^3x \left[\frac{1}{4e^2}\mathcal{F}_{\mu\nu}^2 + \frac{\eta^2}{2}\left(\partial_\mu\left(\theta+\bar\theta\right) - A_\mu\right)^2\right]\right\}. \qquad (40)$$

Here, the full field-strength tensor again reads $\mathcal{F}_{\mu\nu} = F_{\mu\nu} + F_{\mu\nu}^M$, where the monopole part obeys the equation (26) with the replacement $\rho \to \rho_{\rm gas}$. Making use of the relation (13) (where $L$ are now open lines of magnetic vortices, ending at monopoles and antimonopoles), one can perform the path-integral duality transformation of the partition function (40), which yields

$$\mathcal{Z}_{\rm 3D\,AHM} = \int D\varphi Dy_\mu(\tau) Dh_\mu \times \exp\left\{-\int {\rm d}^3x \left[\frac{1}{4\eta^2}\left(\partial_\mu h_\nu - \partial_\nu h_\mu\right)^2 - 2\pi {\rm i} h_\mu \delta_\mu + \left(\frac{e}{\sqrt2}h_\mu + \partial_\mu\varphi\right)^2 + \frac{\rm i}{e\sqrt2}\left(\frac{e}{\sqrt2}h_\mu + \partial_\mu\varphi\right) \times \frac{\partial}{\partial x_\mu}\int {\rm d}^3y \frac{\rho_{\rm gas}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\right]\right\}$$

(*cf.* (15)). Performing now the gauge transformation, which eliminates the field $\varphi$, and integrating over the field $h_\mu$, we obtain

$$\mathcal{Z}_{\rm 3D\,AHM} = \int Dy_\mu(\tau) \exp\left\{-\frac{\pi\eta^2}{8}\int {\rm d}^3x {\rm d}^3y \frac{{\rm e}^{-m_A|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \times \left[4\pi\left(\frac{1}{m_A^2}\rho_{\rm gas}(\mathbf{x})\rho_{\rm gas}(\mathbf{y}) + \delta_\mu(\mathbf{x})\delta_\mu(\mathbf{y})\right) + \int {\rm d}^3z \frac{\rho_{\rm gas}(\mathbf{x})\rho_{\rm gas}(\mathbf{z})}{|\mathbf{y}-\mathbf{z}|}\right]\right\},$$

where $m_A = e\eta$ stands for the gauge-boson mass. The integral

$$\int {\rm d}^3y \frac{{\rm e}^{-m_A|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|\,|\mathbf{y}-\mathbf{z}|} = \int {\rm d}^3u \frac{{\rm e}^{-m_A|\mathbf{u}|}}{|\mathbf{u}|\,|\mathbf{x}-\mathbf{z}-\mathbf{u}|}$$

can easily be calculated by expanding $1/|\mathbf{x}-\mathbf{z}-\mathbf{u}|$ in Legendre polynomials, and the result reads

$$\frac{4\pi}{m_A^2|\mathbf{x}-\mathbf{z}|}\left(1-{\rm e}^{-m_A|\mathbf{x}-\mathbf{z}|}\right).$$

Taking this into account, we can write down the final expression for the partition function of the 3D AHM with external monopoles, which has the following simple form:

$$\mathcal{Z}_{\rm 3D\,AHM} = \int Dy_\mu(\tau) \times \exp\left\{-\frac{\pi\eta^2}{2}\int {\rm d}^3x {\rm d}^3y \left[\frac{{\rm e}^{-m_A|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\delta_\mu(\mathbf{x})\delta_\mu(\mathbf{y}) + \frac{1}{m_A^2}\frac{\rho_{\rm gas}(\mathbf{x})\rho_{\rm gas}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\right]\right\}. \qquad (41)$$

The first term in square brackets on the R.H.S. of (41) represents again the Biot–Savart interaction between the points of the magnetic vortex (and also interaction between vortices, if we include several ones), which are of Yukawa type, i.e., their Coulomb interactions are screened by the condensate of electric Cooper pairs. On the contrary, the interaction between *external* monopoles, represented by the last term on the R.H.S. of (41) remains to be unscreened.

We now see, that when the gauge-boson mass becomes small (when, for example, the magnetic coupling constant $g = 2\pi/e$ becomes large), the Biot–Savart term can be disregarded w.r.t. the interaction of external monopoles. In this limit,

$$\mathcal{Z}_{\rm 3D\,AHM} \longrightarrow \exp\left[-\frac{\pi}{2e^2}\int {\rm d}^3x {\rm d}^3y \frac{\rho_{\rm gas}(\mathbf{x})\rho_{\rm gas}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\right],$$

which is just the statistical weight of the partition function of the monopole gas (18). Clearly, this result is in agreement with that of the corresponding limiting procedure applied directly to (40).

## 4 Summary

In the present paper, we have addressed two problems. The first of them is the investigation of the relation between confinement in the Abelian-projected SU(2) and SU(3) gluodynamics and the interactions between magnetic monopole currents and electric strings. To study this problem, we cast the partition function of the 4D Abelian-projected SU(2) gluodynamics, which is argued to be just the dual Abelian Higgs model, into the form of the integral over the monopole currents. Besides the quadratic part in these currents, the resulting monopole effective action turned out to contain also a term which described the interaction of a monopole current with the electric ANO string. Then we extended our analysis to the case of the Abelian-projected 4D SU(3) gluodynamics, where the resulting representation turned out to contain three monopole currents linked to two independent string world-sheets in a certain way. Finally, for illustration, we also performed the corresponding calculation in 3D, where the role of the moving string world-sheets is played by the static electric vortex lines, and the found expressions are more transparent.

The second topic studied in this paper, is the investigation of the 3D compact QED and its relation to the SVM and the 3D Abelian Higgs model with external monopoles. Firstly, we have demonstrated that the string representation of 3D compact QED (the so-called confining string theory) is nothing else than the integral over the monopole densities. Secondly, in the weak-field limit of 3D compact QED, we have calculated two coefficient functions, which parametrize the bilocal correlator of the field-strength tensors in the analogous case of SVM. One of them was found by two methods: (i) from the correlator of the monopole densities; and (ii) by virtue of the weak-field limit of the confining string theory. Coincidence of both results thus confirms the consistency of our calculations. By making use of this function, we then obtained the string tension of the Nambu–Goto term and the inverse bare coupling constant of the rigidity term, corresponding to the weak-field effective action of the confining string theory. Those turned out to be nonanalytic in the magnetic coupling constant (i.e., explicitly nonperturbative) and positive and negative, respectively, which is important for the stability of the obtained string effective action. The large-distance asymptotic behaviours of both coefficient functions correspond to the ones parametrizing the bilocal correlator of the field-strength tensors in QCD within the SVM. The obtained asymptotic behaviours are also similar to those of the corresponding functions in the DAHM. Finally, we proved that the latter similarity is not accidental, namely, the 3D compact QED is related to the small gauge-boson mass limit of 3D AHM with external monopoles.

In conclusion, the obtained results shed some light on the mechanisms of confinement in various Abelian-projected theories. Furthermore, they prove the relevance of concepts of the SVM to the description of confinement in the 3D compact QED.

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## Appendix A. Path-integral duality transformation

In this appendix, we shall outline some details of a derivation of (3). Firstly, let us linearize the term $\eta^2/2\left(\partial_\mu\left(\theta+\bar\theta\right)-2gB_\mu\right)^2$ in the exponent on the R.H.S. of (1) and carry out the integral over $\theta$ as follows [5,7,8,12]:

$$\begin{aligned}&\int D\theta\exp\left\{-\frac{\eta^2}{2}\int \mathrm{d}^4x\left(\partial_\mu\left(\theta+\bar\theta\right)-2gB_\mu\right)^2\right\}\\&=\int DC_\mu D\theta\exp\left\{\int\mathrm{d}^4x\left[-\frac{1}{2\eta^2}C_\mu^2\right.\right.\\&\left.\left.\quad+\mathrm{i}C_\mu\left(\partial_\mu\left(\theta+\bar\theta\right)-2gB_\mu\right)\right]\right\}\\&=\int DC_\mu\delta\left(\partial_\mu C_\mu\right)\exp\left\{\int\mathrm{d}^4x\left[-\frac{1}{2\eta^2}C_\mu^2\right.\right.\\&\left.\left.\quad+\mathrm{i}C_\mu\left(\partial_\mu\bar\theta-2gB_\mu\right)\right]\right\}.\end{aligned}\quad\text{(A.1)}$$

Next, we can solve the constraint $\partial_\mu C_\mu=0$ by representing $C_\mu$ in the form $C_\mu=\partial_\nu\tilde h_{\mu\nu}\equiv(1/2)\varepsilon_{\mu\nu\lambda\rho}\partial_\nu h_{\lambda\rho}$, where $h_{\lambda\rho}$ stands for an antisymmetric tensor field.

Note that the field $C_\mu$ is related to the monopole current (4) as $C_\mu=-(1/g)j_\mu$. This means that the $\delta$ function in the last equality on the R.H.S. of (A.1) just expresses the conservation of this current.

Next, taking into account the relation (2) between $\bar\theta$ and $\Sigma_{\mu\nu}$, we get

$$\begin{aligned}&\int D\bar\theta D\theta\exp\left\{-\frac{\eta^2}{2}\int\mathrm{d}^4x\left(\partial_\mu\left(\theta+\bar\theta\right)-2gB_\mu\right)^2\right\}\\&=\int Dx_\mu(\xi)Dh_{\mu\nu}\exp\left\{\int\mathrm{d}^4x\left[-\frac{1}{12\eta^2}H_{\mu\nu\lambda}^2\right.\right.\\&\left.\left.\quad+\mathrm{i}\pi h_{\mu\nu}\Sigma_{\mu\nu}-\mathrm{i}g\varepsilon_{\mu\nu\lambda\rho}B_\mu\partial_\nu h_{\lambda\rho}\right]\right\}.\end{aligned}\quad\text{(A.2)}$$

In the derivation of (A.2), we have replaced $D\bar\theta$ by $Dx_\mu(\xi)$, discarding for simplicity the Jacobian [9] arising during such a change of the integration variable.

Bringing together (1) and (A.2), we obtain

$$\begin{aligned}\mathcal{Z}_{4\mathrm{D\,DAHM}}&=\int DB_\mu Dx_\mu(\xi)Dh_{\mu\nu}\\&\times\exp\left\{-\int\mathrm{d}^4x\left[\frac{1}{12\eta^2}H_{\mu\nu\lambda}^2+\frac14F_{\mu\nu}^2\right.\right.\\&\left.\left.-\mathrm{i}\pi h_{\mu\nu}\Sigma_{\mu\nu}+\mathrm{i}g\tilde F_{\mu\nu}h_{\mu\nu}\right]\right\}.\end{aligned}\quad\text{(A.3)}$$

Let us now integrate over the field $B_\mu$. To this end, we find it convenient to rewrite

$$\begin{aligned}&\exp\left(-\frac14\int d^4xF_{\mu\nu}^2\right)\\&=\int DG_{\mu\nu}\exp\left\{\int\mathrm{d}^4x\left[-G_{\mu\nu}^2+\mathrm{i}\tilde F_{\mu\nu}G_{\mu\nu}\right]\right\},\end{aligned}$$

after which the $B_\mu$ integration yields

$$\int DB_\mu \exp\left\{-\int \mathrm{d}^4x \left[\frac{1}{4}F_{\mu\nu}^2 + \mathrm{i}g\tilde{F}_{\mu\nu}h_{\mu\nu}\right]\right\}$$
$$= \int DG_{\mu\nu} \exp\left(-\int \mathrm{d}^4x G_{\mu\nu}^2\right) \delta\left(\varepsilon_{\mu\nu\lambda\rho}\partial_\mu\left(G_{\lambda\rho} - gh_{\lambda\rho}\right)\right)$$
$$= \int DA_\mu \exp\left[-\int \mathrm{d}^4x \left(gh_{\mu\nu} + \partial_\mu A_\nu - \partial_\nu A_\mu\right)^2\right]. \quad \text{(A.4)}$$

Here, $A_\mu$ is just the usual gauge field, dual to the dual vector potential $B_\mu$. Finally, by substituting (A.4) into (A.3), we arrive at (3) of the main text.

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